Exercise sheet 1: Special relativity and tensor calculus

Please prepare your solutions to the following problems, ready to present in the class on **27.04.2022** at **16:00**. In these problems we always assume, unless otherwise stated, a $\{-,+,+,+\}$ metric signature, and we choose units so that the speed of light is $c = 1$.

1. A cart rolls straight across a table with speed v with respect to the table. On the cart is another cart, rolling with speed u with respect to the first cart, in the same direction.
   (a) What is the speed of the second cart with respect to the table?
   (b) Now assume $u = v$. On the second cart there is a third cart moving at speed v with respect to the second, and in the same direction. On the third cart there is a fourth cart moving at speed v relative to the third, in the same direction. And so on, up to an n-th cart.
      i. What is the speed v_n of the n-th cart with respect to the table?
      ii. What is the limit $\lim_{n\to\infty} v_n$?
2. Consider a quasar that ejects gas with speed v at angle θ with respect to the line of sight of an observer on Earth. Projected onto the sky, the gas appears to travel perpendicular to the line of sight, with angular speed v_{app}/D, where D is the distance to the quasar, and v_{app} is the apparent speed.
   (a) Derive an expression for v_{app} in terms of v and θ.
   (b) Show that there are appropriate values of v and θ so that $v_{\text{app}} > 1$.
3. Suppose X and Y are rank $(0,3)$ tensors related via
   $$X_{\gamma\alpha\beta} + X_{\beta\alpha\gamma} = Y_{\alpha\beta\gamma}.$$
   Suppose further that X is symmetric in its latter two indices (i.e. $X_{\alpha\beta\gamma} = X_{\alpha(\beta\gamma)}$).
   (a) Write $X_{\alpha\beta\gamma}$ solely in terms of the tensor Y.
   (b) How does your answer to part (a) change if X is *anti*symmetric in its latter two indices (i.e. $X_{\alpha\beta\gamma} = X_{\alpha[\beta\gamma]}$)?
4. Prove the following about four-vectors v^μ and w^μ in Minkowski space:
   (a) If v^μ is timelike and $v^\mu w_\mu = 0$, then w^μ is spacelike.
   (b) If both v^μ and w^μ are timelike and $v^\mu w_\mu < 0$, then either both are future-pointing or both are past-pointing.
   (c) If v^μ and w^μ are null and $v^\mu w_\mu = 0$, then v^μ is proportional to w^μ.
   (d) If v^μ is null and $v^\mu w_\mu = 0$, then either w^μ is proportional to v^μ, or w^μ is spacelike.